UDC 539.3

COMPRESSION OF AN ELASTIC LAYER BY GIRDER PLATES

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The exact solution of the contact problem of the action of a semi-infinite girder on an elastic strip is used to construct a corresponding system of piecewise homogeneous solutions which are then used to study the problems of compression of an elastic strip by a periodic succession of identical finite girders. The arbitrary constants appearing in the piecewise homogeneous spolutions are determined from an infinite system of algebraic equations possessing a normal. two-sided determinant with exponentially decreasing elements.

The problem of action of a semi-infinite girder plate on a linearly deformable foundation and, in particular, on an elastic strip, has been first solved by Popov [1, 2] who however used a different method.

1. Let us consider a semi-infinite girder plate $x \geqslant 0,\ y = 1$ of constant thickness h resting on an elastic layer $0 \leqslant y \leqslant 1$, which in turn lies on a perfectly rigid, smooth foundation. Assume that the load $q(x)$ on the plate, the transverse force P and moment M applied to its edge, and the normal extra load $r(x)$ acting on the free part $x < 0$ of the layer, do not vary in the direction of the girder plate edge and, that the friction on both surfaces of the layer is zero. Then the elastic layer will undergo plane deformation and the boundary conditions of the problem for a corresponding infinite strip $0 \leqslant y \leqslant 1$ compressed by a semi-infinite girder $x \geqslant 0,\ y = 1$ have the form

$$\tau_{xy} = v = 0\ (y = 0),\ \tau_{xy} = 0\ (y = 1) \tag{1.1}$$

$$\sigma_y = r(x)\ (y = 1,\ x < 0),\ \eta(x) \equiv D\partial^4 v / \partial x^4 + \sigma_y = q(x)\ (y = 1,\ x \geqslant 0) \tag{1.2}$$

$$D = {}^1/_{12} E_0 h^3 (1 - \nu_0^2)^{-1}$$

where D is the rigidity, E_0 is the modulus of elasticity and ν_0 is the Poisson's ratio for the girder.

We seek a solution of this problem in the form of Papkovich-Neuber

$$u(x,\ y) = F_2 - \frac{1}{4(1-\nu)} \frac{\partial}{\partial x} (yF_1 + xF_2 + F_3) \tag{1.3}$$

$$v(x,\ y) = F_1 - \frac{1}{4(1-\nu)} \frac{\partial}{\partial y} (yF_1 + xF_2 + F_3)$$

Let us set $F_2 = 0,\ F_1 = \partial F_4 / \partial y,\ F_3 = 4(1 - \nu)(F_4 - F_5)$ where F_4 and F_5 are harmonic functions, and apply the two-sided Laplace transform to the expressions (1.3). Taking the conditions (1.1) into account, we obtain

$$u(p, y) = \int_{-\infty}^{\infty} u(x,\ y)\, e^{-px} dx = C(p)\, p\, [\varepsilon(p) - \rho(p)] \tag{1.4}$$

$$v(p,\ y) = \int_{-\infty}^{\infty} v(x,\ y)\, e^{-px} dx = C(p)\, [\varepsilon'(p) + \rho'(p)]$$

$$\rho(p) = 2(1-\nu)\sin p \cos py,\ \varepsilon(p) = \sin p \cos py + p(\cos p \cos py + y \sin p \sin py)$$

Here the prime denotes the derivative with respect to y and the function $C(p)$ satisfies, by virtue of the conditions (1.2), the equations

$$\sigma^+(p) + \sigma^-(p) = -C(p)p^2 \frac{EN_1(p)}{2(1+\nu)}, \quad \eta^+(p) + \eta^-(p) = -C(p)p^2 \frac{EN_2(p)}{2(1+\nu)} \tag{1.5}$$

where

$$N_1(p) = \sin 2p + 2p,\ N_2(p) = 2ap^3 \sin^2 p + \sin 2p + 2p,\ a = 2(1-\nu^2)E^{-1}D$$

$$\sigma^+(p) = \int_0^\infty \sigma_y(x, 1)e^{-px}dx, \quad \sigma^-(p) = \int_{-\infty}^0 r(x)e^{-px}dx$$

$$\eta^+(p) = \int_0^\infty q(x)e^{-px}dx, \quad \eta^-(p) = \int_{-\infty}^0 \eta(x)e^{-px}dx$$

The functions $N_1(p)$ and $N_2(p)$ both are odd. As the following inequalities show, they have no real and no pure imaginary zeros except $p = 0$:

$$2ap^3\sin^2 p + \sin 2p + 2p > \sin 2p + 2p > 0 \quad \text{for } p > 0$$

$$2ap^3 \operatorname{sh}^2 p + \operatorname{sh} 2p + 2p > \operatorname{sh} 2p + 2p > 0 \quad \text{for } p > 0$$

Let us denote the complex zeros of the functions $N_1(p)$ and $N_2(p)$ in the quadrant $\operatorname{Re} p > 0$, $\operatorname{Im} p > 0$ by a_k and b_k $(k = 1, 2, \ldots)$, respectively, and introduce the numbers $a_k = -a_{-k}$ and $b_{-k} = -b_k$. The following estimates hold for large k :

$$a_k = k\pi + iO(\ln k) + O(1),\ b_k = k\pi + o(1) \tag{1.6}$$

By the Laplace inversion theorem we have

$$u = \frac{1}{2\pi i}\int_L C(p)p[\varepsilon(p) - \rho(p)]e^{px}dp, \quad v = \frac{1}{2\pi i}\int_L C(p)[\varepsilon'(p) + \rho'(p)]e^{px}dp$$

$$\sigma_y = -\frac{E}{2\pi i(1+\nu)}\int_L C(p)p^2\varepsilon(p)e^{px}dp, \quad \sigma_x = -\frac{E}{2\pi i(1+\nu)}\int_L C(p)\varepsilon''(p)e^{px}dp$$

$$\tau_{xy} = \frac{E}{2\pi i(1+\nu)}\int_L C(p)p\varepsilon'(p)e^{px}dp \tag{1.7}$$

The domain of variation of the parameter p and the path of integration L passing through this domain both depend on the character of the functions $r(x)$ and $q(x)$, and on the form of the displacements and stresses under investigation. Suppose that the loads $q(x)$ and $r(x)$ are local or decrease exponentially when $|x| \to \infty$. Then the function $u(p, y)$ exists in the strip $0 < \operatorname{Re} p < \alpha$, while the transforms of the displacement $v(x, y)$ and of all stresses, as well as the functions appearing in (1.5), all exist in the strip $|\operatorname{Re} p| < \alpha$ and in particular on the imaginary axis.

Thus the Wiener-Hopf equation [3]

$$\sigma^+(p) = K(p)\eta^-(p) + K(p)\eta^+(p) - \sigma^-(p) \tag{1.8}$$

obtained by eliminating the functions $C(p)$ from (1.5) can be solved by treating it as a problem of linear conjugation [4] on the imaginary axis, with the coefficient $K(p) =$

$N_1(p) / N_2(p)$ and the free term $K(p)\eta^+(p) - \sigma^-(p)$.

First we consider the homogeneous problem

$$\sigma^+(p) = K(p)\eta^-(p) \tag{1.9}$$

obtained when the forces are applied to the end-face of the girder only. We write (1.9) in the form

$$X^+(p) = G(p) X^-(p) \tag{1.10}$$

$$X^+(p) = a^{1/2}(1-p)^{3/2}[\eta^-(p)]^{-1}, \quad X^-(p) = a^{-1/2}(1+p)^{-3/2}[\sigma^+(p)]^{-1}$$

$$G(p) = a(1-p^2)^{3/2}K(p), \quad -\pi < \arg(1 \pm p) < \pi$$

The function $G(p)$ which satisfies the Hölder condition over the whole contour, has the power index $\varkappa = 0$. Following F. D. Gakhov, we can write the solution of the problem (1.10) in the form

$$X(p) = \exp\left\{\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\ln G(t)\,dt}{t-p}\right\}$$

$$X^{\pm}(i\tau) = \exp\left\{\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\ln G(t)\,dt}{t-i\tau} \pm \frac{1}{2}\ln G(i\tau)\right\}$$

Returning to the initial notation and taking into account the parity of the function $K(p)$, we obtain the following canonical solution of (1.9)

$$\sigma_0^+(p) = a^{-1/2}(1+p)^{-3/2}\exp\left\{\frac{p}{\pi}\int_0^\infty \frac{\ln[a(1+t^2)^{3/2}K(it)]\,dt}{t^2+p^2}\right\} \tag{1.11}$$

$$\eta_0^-(p) = [\sigma_0^+(-p)]^{-1} \tag{1.12}$$

and we have

$$\sigma^+_0(p) \sim a^{-1/2}p^{-3/2} \text{ when } p \to \infty, \ \mathrm{Re}\, p \geqslant 0 \tag{1.13}$$

On the imaginary axis we have

$$\sigma_0^+(i\tau) = (1+i\tau)^{-3/2}(1+\tau^2)^{3/4}[K(i\tau)]^{1/2}\exp\left\{\frac{i\tau}{\pi}\int_0^\infty \ln\left[\frac{(1+t^2)^{3/2}K(it)}{(1+\tau^2)^{3/2}K(i\tau)}\right]\frac{dt}{t^2-\tau^2}\right\} \tag{1.14}$$

The general solution of (1.8) with $r(x) = 0$ has the form

$$\sigma^+(p) = -\frac{\sigma_0^+(p)}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\eta^+(t)\,dt}{\eta_0^-(t)(t-p)} + \sigma_0^+(p)(Ap+B) \tag{1.15}$$

and on the imaginary axis we have

$$\sigma^+(i\tau) = \frac{\sigma_0^+(i\tau)}{2\pi i}\int_{-i\infty}^{i\infty}\left[\frac{\eta^+(i\tau)}{\eta_0^-(i\tau)} - \frac{\eta^+(t)}{\eta_0^-(t)}\right]\frac{dt}{t-i\tau} + \frac{1}{2}K(i\tau)\eta^+(i\tau) + \sigma_0^+(i\tau)[Ai\tau+B] \tag{1.16}$$

From (1.15) and (1.13) it follows that if $|\eta^+(it)| \sim t^{1/2-\varepsilon}$ $(\varepsilon > 0)$ when $t \to \infty$, then $\sigma^+(p) \sim p^{-5/2}$ when $A = B = 0$ and $p \to \infty$. Using this estimate and the corresponding proof from [5] we can conclude, that the character of the normal stresses under the edge of an arbitrarily loaded girder, is described by the formula

$$\sigma_y(x, 1) = A(\pi a x)^{-1/2} + BO(x^{1/2}) + O(x^{3/2}) \text{ when } x \to +0 \tag{1.17}$$

in which the constants A and B can be obtained from the condition of equilibrium. Let the moment M act in the anticlockwise direction on the unit end-face of the girder plate and the transverse force P in the direction of the y-axis. Then

$$P = \int_0^\infty \sigma_y(x, 1)\,dx - \eta^+(0), \qquad M = \int_0^\infty \sigma_y(x, 1)\,x dx + \eta^{+*}(0) \tag{1.18}$$

where the asterisk denotes the derivative with respect to p. Let us now substitute the following expression into condition (1.18):

$$\sigma_y(x, 1) = \frac{1}{2\pi i}\int_L \sigma^+(p)\,e^{px}dp$$

in which the contour L has been displaced to the left past the imaginary axis, and the order of integration has been changed. Integrating with respect to x and completing the contour L from the right with a semicircle of a large radius, using the estimate (1.13) and the fact that the function $\sigma^+(p)$ is regular in the right semiplane, we obtain by virtue of the theorem of residues, the following:

$$P = \sigma^+(0) - \eta^+(0), \quad M = -\sigma^{+*}(0) + \eta^{+*}(0)$$

Hence, taking into account the relations

$$\sigma_0^+(0) = \eta_0^-(0) = K(0) = 1, \quad K^*(0) = 0$$

and using the formulas (1.12), (1.14) and (1.16) as well as their differentials with respect to p, we obtain after some manipulations

$$A = -M - \sigma_0^{+*}(0)\left[P + \frac{1}{2}\eta^+(0)\right] + \frac{1}{2}\eta^{+*}(0) + \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\left\{\left[\frac{\eta^+(t)}{\eta_0^-(t)}\right]^* - \left[\frac{\eta^+(0)}{\eta_0^-(0)}\right]^*\right\}\frac{dt}{t}$$

$$B = P + \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\left[\frac{\eta^+(t)}{\eta_0^-(t)} - \eta^+(0)\right]\frac{dt}{t} + \frac{\eta^+(0)}{2} \tag{1.19}$$

$$\eta_0^{-*}(i\tau) = \eta_0^-(i\tau)\left\{\frac{i}{\pi}\int_0^\infty\left[\frac{K^*(it)}{K(it)} - \frac{K^*(i\tau)}{K(i\tau)}\right]\frac{tdt}{t^2-\tau^2} - \frac{K^*(i\tau)}{2K(i\tau)}\right\}$$

$$\sigma_0^{+*}(0) = \frac{i}{\pi}\int_0^\infty \frac{K^*(it)\,dt}{K(it)\,t}$$

In particular, when $q(x) = 0$ we have

$$A = -M - \sigma_0^{+*}(0)\,P, \quad B = P \tag{1.20}$$

Let us discuss the types of load which we will need on Sects. 2 and 3. We assume $r(x) = 0$, $q(x) = q_k e^{a_k x}$ then $\eta^+(p) = q_k(p - a_k)^{-1}$. Completing the contour of integration in (1.15) from the left by a semicircle of a large radius and using the Jordan lemma together with the theorem of residues, we obtain

$$\sigma^+(p) = \sigma_0^+(p)\,\{[(p - a_k)\,\eta_0^-(a_k)]^{-1}q_k + Ap + B\} \tag{1.21}$$

If $r(x) = r_k e^{b_k x}$ and $q(x) = 0$, then $\sigma^-(p) = r_k(b_k - p)^{-1}$ and the Liouville's theorem applied directly to (1.8) yields

$$\sigma^+(p) + \sigma^-(p) = \sigma_0^+(p)\,\{r_k[(b_k - p)\,\sigma_0^+(b_k)]^{-1} + Ap + B\} \tag{1.22}$$

Let a concentrated force $(-Q)$ act on the girder at the point $x = c$. In this case the solution of the problem is expressed by (1.15) in which the substitution $\eta^+(p) = -Qe^{-cp}$ must be made.

Let us construct the solution in a different form. We separate the problem (1.1) and (1.2) with the conditions $r(x) = 0$, $P = M = 0$ and $q(x) = -Q\delta(x - c)$, where $\delta(x)$ is the Dirac delta function, into two parts, the fundamental problem

$$\tau_{xy} = v = 0 \quad \text{at} \quad y = 0, \ \tau_{xy} = 0 \quad \text{at} \quad y = 1$$

$$\eta(x) = -Q\delta(x - c) \quad \text{at} \ y = 1 \tag{1.23}$$

and the mixed problem consisting of (1.1) and (1.2) with the conditions

$$q(x) = 0, \quad r(x) = -\sigma_y^{(\delta)}(x, 1), \quad P = -P^{(\delta)}, \quad M = -M^{(\delta)} \tag{1.24}$$

Here $\sigma_y^{(\delta)}(x, 1)$ denotes the normal stress in the strip at $y = 1$, while $P^{(\delta)}$ and $M^{(\delta)}$ are the transverse force and moment in the solution of (1.23) at the point $y = 1$, $x = 0$. This solution is obviously given by the formulas (1.7) and we have

$$C(p) = \frac{2Q(1+\nu)e^{-pc}}{Ep^2N_2(p)} \tag{1.25}$$

Expanding the integral in the expression for σ_y in (1.7) into a series in terms of residues expressed by the zeros of $N_2(p)$ and integrating with respect to x from $-\infty$ to 0, we obtain

$$\sigma_y^{(\delta)}(x, 1) = \operatorname{Re}\sum_{k=1}^{\infty} c_k e^{b_k x}, \qquad P^{(\delta)} = -\operatorname{Re}\sum_{k=1}^{\infty} c_k b_k^{-1}$$

$$M^{(\delta)} = \operatorname{Re}\sum_{k=1}^{\infty} c_k b_k^{-2}, \qquad c_k = 2Qe^{-b_k c} N_1(b_k)\,[N_2^*(b_k)]^{-1} \tag{1.26}$$

Solving the problem (1.1), (1.2), (1.24) and (1.26) by means of the formula (1.22) and adding the solution (1.7), (1.25), we obtain

$$u = QH\{p[\varepsilon(p) - \rho(p)], c, x\}, \ v = QH\{\varepsilon'(p) + \rho'(p), c, x\}$$

$$H\{f(p), c, x\} = \frac{1+\nu}{\pi i E}\int_L \left\{\frac{e^{-pc}}{p^2N_2(p)} - \frac{\sigma_0^+(p)}{N_1(p)}\sum_{k=1}^{\infty}[t_1(b_k, p) + t_1(\bar{b}_k, p)]\right\} f(p)\, e^{px}\, dp$$

$$t_m(\tau, p) = \frac{N_1(\tau)\, r_m(\tau)}{\tau^2(p - \tau)\,\sigma_0^+(\tau)\, N_2^*(\tau)}, \qquad r_1(\tau) = e^{-\tau c} \tag{1.27}$$

In the same manner we construct a Green function for the problem of additional load. However in this case we cannot apply the Gakhov's method directly because when $r(x) \neq 0$, the free term in (1.8) does not satisfy the Hölder boundary condition.

2. Let us construct a system of piecewise-homogeneous solutions, setting in the conditions (1.1) and (1.2) $q(x) = r(x) = 0$ and assuming that the girder end is load-free. Following [5] we shall consider the subsystems with singularities at the points $x = \infty$ and $x = -\infty$ separately.

The homogeneous boundary conditions

$$v = \tau_{xy} = 0 \quad (y = 0), \qquad \eta(x) = \tau_{xy} = 0 \quad (y = 1)$$

are satisfied by the solution

$$u^{k1}(x, y) = \Phi^{k1}\{p[\varepsilon(p) - \rho(p)]\},\quad v^{k1}(x, y) = \Phi^{k1}\{\varepsilon'(p) + \rho'(p)\}$$

$$\Phi^{k1}\{f(p)\} = A_k \operatorname{Re}[f(b_k)e^{b_k x}] + B_k \operatorname{Im}[f(b_k)e^{b_k x}]$$

where A_k and B_k are arbitrary constants and $k = 1, 2, \ldots$.

Supplementing this with the solution of the mixed problem (1.1), (1.2) with the conditions

$$q(x) = 0, \qquad r(x) = -\sigma_y^{k1}(x, 1)$$

$$P = -\frac{E}{2(1+\nu)}\{A_k \operatorname{Re}[b_k N_1(b_k)] + B_k \operatorname{Im}[b_k N_1(b_k)]\},$$

$$M = \frac{E}{2(1+\nu)}[A_k \operatorname{Re} N_1(b_k) + B_k \operatorname{Im} N_1(b_k)]$$

which, by virtue of the formulas (1.7), (1.5) and (1.22) have the form

$$u^{k2}(x, y) = \Phi^{k2}\{p[\varepsilon(p) - \rho(p)]\}, \qquad v^{k2}(x, y) = \Phi^{k2}\{\varepsilon'(p) + \rho'(p)\}$$

$$\Phi^{k2}\{f(p)\} = A_k \operatorname{Re} H_{k2}[f(p), x] + B_k \operatorname{Im} H_{k2}[f(p), x]$$

$$H_{k2}[f(p), x] = \frac{N_1(b_k)}{2\pi i \sigma_0^+(b_k)} \int_L \frac{\sigma_0^+(p) f(p) e^{px} dp}{(p - b_k) N_1(p)}.$$

we obtain the first subsystem ($k = 1, 2, \ldots$)

$$u^{(k)}(x, y) = \Phi^{(k)}\{p[\varepsilon(p) - \rho(p)]\},\quad v^{(k)}(x, y) = \Phi^{(k)}\{\varepsilon'(p) + \rho'(p)\} \tag{2.1}$$

$$\Phi^{(k)}\{f(p)\} = A_k \operatorname{Re} H_k[f(p), x] + B_k \operatorname{Im} H_k[f(p, x)],$$
$$H_k[f(p), x] = f(b_k)e^{b_k x} + H_{k2}[f(p), x]$$

The second subsystem with a singularity at $x = -\infty$, is constructed in the analogous manner from the solution of (1.21), and has the form ($k = -1, -2, \ldots$)

$$u^{(k)}(x, y) = \Phi^{(k)}\{p[\varepsilon(p) - \rho(p)]\}, \qquad v^{(k)}(x, y) = \Phi^{(k)}\{\varepsilon'(p) + \rho'(p)\}$$

$$\Phi^{(k)}\{f(p)\} = A_k \operatorname{Re} H_k[f(p), x] + B_k \operatorname{Im} H_k[f(p), x] \tag{2.2}$$

$$H_k[f(p), x] = f(a_k)e^{a_k x} - \frac{N_2(a_k)}{2\pi i \eta_0^-(a_k)} \int_L \frac{\sigma_0^+(p) f(p) e^{px} dp}{(p - a_k) N_1(p)}$$

The elements of both subsystems are self-balanced. A solution determining uniform compression and rigid displacement of the strip under the girder has the form

$$u^{(0)}(x, y) = A_0 E^{-1} x + B_0,\quad v^{(0)}(x, y) = -A_0 \nu (1 - \nu)^{-1} E^{-1} y \tag{2.3}$$

The system (2.1) – (2.3) can be used to solve the problems in which an elastic strip or a rectangle is acted upon by several, arbitrarily loaded finite girders, and for solving various periodic type problems with several girders per period. The problem of determining the coefficients A_k and B_k can always be reduced to that of solving normal systems of algebraic equations containing infinite matrix elements the number of rows and columns in which decreases exponentially.

3. As an example, let us consider the following problem. A number of identical girders of length 2λ and hight h are periodically distributed over an elastic strip placed on a plane, nondeformable base. The distance between the adjacent ends of the neighboring girders is equal to 2μ, there is no friction at the strip boundaries, and two concentrated forces $(-Q_1)$ are applied symmetrically to each girder, at the distance c from each end.

Let us connect the point $x = 0$, $y = 1$ with the left end of any one girder, retaining the previous notation for the elastic constants. Then by virtue of periodicity and symmetry, we have the followng problem for an elastic rectangle:

$$v = \tau_{xy} = 0 \ (y = 0, \ -\mu \leqslant x \leqslant \lambda), \ \tau_{xy} = 0 \ (y = 1, \ -\mu \leqslant x \leqslant \lambda) \tag{3.1}$$

$$\sigma_y = 0 \ (y = 1, \ -\mu \leqslant x < 0), \ \eta(x) = -Q_1\delta(x - c) \ (y = 1, \ 0 \leqslant x \leqslant \lambda) \tag{3.2}$$

$$u = \tau_{xy} = 0 \ (x = \lambda, \ x = -\mu, \ 0 \leqslant y \leqslant 1) \tag{3.3}$$

$$\partial v / \partial x = \partial^3 v / \partial x^3 = 0 \ (y = 1, \ x = \lambda) \tag{3.4}$$

We shall seek its solution in the form

$$u = u^{(\delta)} + \sum_{k=-\infty}^{\infty} u^{(k)}, \qquad v = v^{(\delta)} + \sum_{k=-\infty}^{\infty} v^{(k)} \tag{3.5}$$

where $u^{(\delta)}$ and $v^{(\delta)}$ represent the solution of the problem (1.1), (1.2) and are given by (1.27) with the following boundary conditions

$$P = M = 0, \ r(x) = 0$$

$$q(x) = -Q_1\delta(x - c) - Q_2\delta(x + c - 2\lambda)$$

We note that the application to the girder of an additional arbitrary force Q_2 outside the rectangle, satisfies effectively the second boundary condition (3.4). The first condition follows automatically from (3.3). Expanding the function $u^{(\delta)}$ at the rectangle ends into series in residues expressed in terms of the zeros of $N_2(p)$ for $x = \lambda$ and of the zeros of $N_1(p)$ for $x = -\mu$ we obtain, by virtue of (1.27),

$$u^{(\delta)}(\lambda, y) = \sum_{m=1}^{2} Q_m \sum_{k=-1}^{-\infty} [t_{m1}(b_k)\chi(b_k) + t_{m1}(\bar{b}_k)\chi(\bar{b}_k)] + t_0$$

$$u^{(\delta)}(-\mu, y) = \sum_{m=1}^{2} Q_m \sum_{k=1}^{\infty} [t_{m2}(a_k)\chi(a_k) + t_{m2}(\bar{a}_k)\chi(\bar{a}_k)] \tag{3.6}$$

$$t_{m1}(p) = \frac{4(1+\nu)e^{p\lambda}}{EN_2^*(p)}\left\{\frac{t_m(p)}{p^1} - \eta_0^-(p)\sum_{s=1}^{\infty}[t_m(b_s, p) + t_m(\bar{b}_s, p)]\right\}$$

$$t_{m2}(p) = \frac{4(1+\nu)\sigma_0^+(p)}{EN_1^*(p)e^{p\mu}}\sum_{s=1}^{\infty}[t_m(b_s, p) + t_m(\bar{b}_s, p)]$$

$$r_1(\tau) = e^{-\tau c}, \qquad r_2(\tau) = e^{-\tau(2\lambda - c)}, \qquad \chi(p) = {}^1/_2\, p[\varepsilon(p) - \rho(p)]$$

$$t_1(p) = -t_2(p) = e^{-pc}, \qquad t_0 = \nu(1+\nu)E^{-1}\sum_{m=1}^{2} Q_m t_m(0)$$

The latter and the conditions (3.3) together imply that $A_0 = -t_0 E(\lambda + \mu)^{-1}$ and $B_0 = -t_0\mu(\lambda + \mu)^{-1}$. Writing the displacements $u^{(k)}$ given by (2.1) and (2.2) also in the form of series in residues we obtain, for $k = 1, 2, \ldots$

$$u^{(k)}(\lambda, y) = (A_k - iB_k)\Big\{e^{b_k\lambda}\chi(b_k) + \sum_{n=-1}^{-\infty}[T_1(b_k, b_n)e^{b_n\lambda}\chi(b_n) + T_1(b_k, \bar{b}_n)e^{\bar{b}_n\lambda}\chi(\bar{b}_n)]\Big\} + (A_k + iB_k)\Big\{e^{\bar{b}_k\lambda}\chi(\bar{b}_k) + \sum_{n=-1}^{-\infty}[T_1(\bar{b}_k, b_n)e^{b_n\lambda}\chi(b_n) + T_1(\bar{b}_k, \bar{b}_n)e^{\bar{b}_n\lambda}\chi(\bar{b}_n)]\Big\} \tag{3.7}$$

$$u^{(k)}(-\mu, y) = \sum_{n=1}^{\infty}\{(A_k - iB_k)[T_2(b_k, a_n)e^{-a_n\mu}\chi(a_n) + T_2(b_k, \bar{a}_n)e^{-\bar{a}_n\mu}\chi(\bar{a}_n)] + (A_k + iB_k)[T_2(\bar{b}_k, a_n)e^{-a_n\mu}\chi(a_n) + T_2(\bar{b}_k, \bar{a}_n)e^{-a_n\mu}\chi(\bar{a}_n)]\}$$

$$T_1(\tau, p) = \frac{N_1(\tau)\eta_0^-(p)}{(p-\tau)\sigma_0^+(\tau)N_2^*(p)}, \qquad T_2(\tau, p) = -\frac{N_1(\tau)\sigma_0^+(p)}{(p-\tau)\sigma_0^+(\tau)N_1^*(p)}$$

and for $k = -1, -2, \ldots$

$$u^{(k)}(\lambda, y) = \sum_{n=-1}^{-\infty}\{(A_k - iB_k)[T_3(a_k, b_n)e^{b_n\lambda}\chi(b_n) + T_3(a_k\bar{b}_n)e^{\bar{b}_n\lambda}\chi(\bar{b}_n)] + (A_k + iB_k)[T_3(\bar{a}_k, b_n)e^{b_n\lambda}\chi(b_n) + T_3(\bar{a}_k, \bar{b}_n)e^{\bar{b}_n\lambda}\chi(\bar{b}_n)]\} \tag{3.8}$$

$$u^{(k)}(-\mu, y) = (A_k - iB_k)\Big\{e^{-a_k\mu}\chi(a_k) + \sum_{n=1}^{\infty}[T_4(a_k, a_n)e^{-a_n\mu}\chi(a_n) + T_4(a_k, \bar{a}_n)e^{-a_n\mu}\chi(\bar{a}_n)]\Big\} + (A_k + iB_k)\Big\{e^{-\bar{a}_k\mu}\chi(\bar{a}_k) + \sum_{n=1}^{\infty}[T_4(\bar{a}_k, a_n)e^{-a_n\mu}\chi(a_n) + T_4(\bar{a}_k, \bar{a}_n)e^{-\bar{a}_n\mu}\chi(\bar{a}_n)]\Big\}$$

$$T_3(\tau, p) = -\frac{N_2(p)\eta_0^-(p)}{(p-\tau)\eta_0^-(\tau)N_2^*(p)}, \qquad T_4(\tau, p) = \frac{N_2(\tau)\sigma_0^+(p)}{(p-\tau)\eta_0^-(\tau)N_1^* p)}$$

Let us now substitute the series (3.6) – (3.8) into the expression (3.5) for the displacement u, at the same time satisfying the conditions (3.3) imposed on u (this is sufficient for the conditions for τ_{xy} also to be satisfied). We change the order of summation in the double sums obtained, and, noting that $\chi(-p) = \chi(p)$, equate the multipliers of the functions $\chi(a_k)$, $\chi(\bar{a}_k)$, $\chi(b_k)$ and $\chi(b_k)$. Introducing the new unknowns

$$X_k - iY_k = (A_k - iB_k)\exp[b_k(^3/_2\lambda - {}^1/_2 c)] \quad \text{for } k \geqslant 1$$

$$X_k - iY_k = (A_k - iB_k)\exp(-a_k\mu) \quad \text{for } k \leqslant -1$$

and separating the real and imaginary parts, we obtain the following infinite system of algebraic equations

$$Q_2 \operatorname{Re}\psi_2(d_k) + X_k + \sum_{n=-\infty}^{\infty}\{X_n \operatorname{Re}[\varphi_n(d_k) + \varphi_n(\bar{d}_k)] + Y_n \operatorname{Im}[\varphi_n(d_k) + \varphi_n(\bar{d}_k)]\} = -Q_1 \operatorname{Re}\psi_1(d_k) \tag{3.9}$$

$$-Q_2 \operatorname{Im}\psi_2(d_k) + Y_k + \sum_{n=-\infty}^{\infty}\{Y_n \operatorname{Re}[\varphi_n(d_k) - \varphi_n(\bar{d}_k)] - X_n \operatorname{Im}[\varphi_n(d_k) - \varphi_n(\bar{d}_k)]\} = Q_1 \operatorname{Im}\psi_1(d_k)$$

where under the summation signs we have $n \neq 0$; $k = \pm 1, \pm 2, \ldots$; $d_k = b_k$ when $k \geqslant 1$ and $d_k = a$ when $k \leqslant -1$; $\psi_m(a_k) = t_{m2}(-a_k)$.

$$\psi_m(b_k) = t_{m1}(-b_k)\exp[{}^1/_2\, b_k(\lambda - c)] \tag{3.10}$$

$$\varphi_n(a_k) = T_2(b_n, -a_k)\exp(\mu a_k - {}^3/_2\,\lambda b_n + {}^1/_2\, c b_n) \qquad (n \geqslant 1)$$

$$\varphi_n(b_k) = T_1(b_n, -b_k)\exp[-\lambda({}^3/_2\, b_n + {}^1/_2\, b_k) + {}^1/_2\, c(b_n - b_k)] \qquad (n \geqslant 1)$$

$$\varphi_n(a_k) = T_4(a_n, -a_k)\exp(\mu a_k + \mu a_n) \qquad (n \leqslant -1)$$

$$\varphi_n(b_k) = T_3(a_n, -b_k)\exp(\mu a_n - {}^1/_2\,\lambda b_k - {}^1/_2\, c b_k) \qquad (n \leqslant -1)$$

Satisfying the second condition of (3.4) we obtain

$$-Q_2 H[2(1-\nu)p^4\sin^2 p,\ 2\lambda - c,\ \lambda] + \tag{3.11}$$

$$\sum_{n=-\infty}^{\infty}(X_n \operatorname{Re}\gamma_n + Y_n \operatorname{Im}\gamma_n) = Q_1 H[2(1-\nu)p^4\sin^2 p,\ c,\ \lambda]$$

in which

$$\gamma_n = -H_n[2(1-\nu)p^4\sin^2 p, \lambda]\exp[-b_n({}^3/_2\lambda - {}^1/_2 c)] \quad \text{for } n \geqslant 1$$

$$\gamma_n = -H_n[2(1-\nu)p^4\sin^2 p, \lambda]\exp(a_n\mu) \quad \text{for } n \leqslant -1 \tag{3.12}$$

Formulas (3.10) and (3.12) show that the nondiagonal elements of the system (3.9), (3.11) decrease exponentially in both the number of rows and the number of columns. Therefore this system can be classed as a normal Poincaré-Koch system [6]. Estimating its solution (see Sect. 2 of [5]) for large k we obtain $A_k, B_k \sim O(|k|^{5/2} e^{-2\pi |k| \mu})$ for $k < 0$ and $A_k, B_k \sim O[k^{-5} e^{-k\pi(2\lambda - c)}]$ for $k > 0$.

We note that in this paper we have solved another two problems. Setting $A_k = B_k = 0$ for $k \leqslant -1$ in the expressions (3.5) and retaining, out of four blocks in the system (3.9) only the block containing the elements with indices $k \geqslant 1$ and $n \geqslant 1$, we obtain a solution of the problem of pressure of a single girder of length 2λ on an elastic strip, while setting $Q_2 = 0$ and $A_k = B_k = 0$ in (3.5), for $k \geqslant 1$, neglecting (3.11) and three superfuous blocks of the system (3.9), we find that the expressions (3.5) represent a solution of the problem of pressure of two semi-infinite girders the ends of which are separated by 2μ.

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